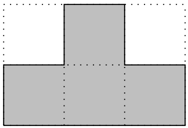
Tesselation and polyomino composition

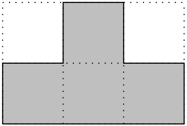
Invariance of exactness under composition

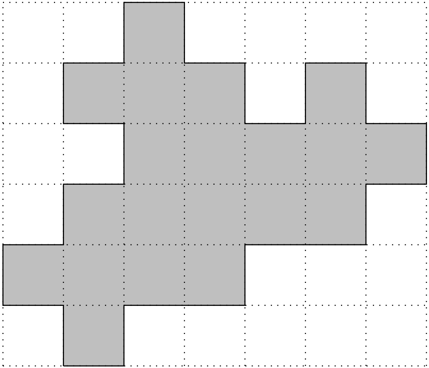
Pierre Cagne
with Sébastien Labbé, Srečko Brlek and Xavier Provençal at UQÀM

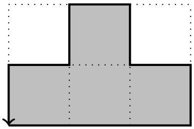
Journée GT Géométrie Discrète

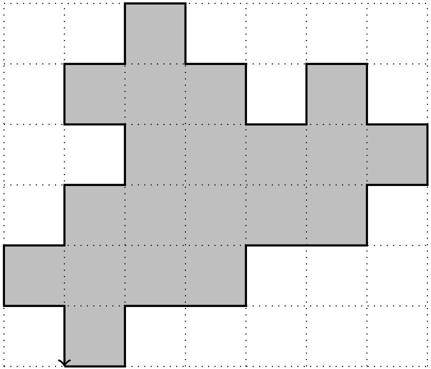
13 juin 2013











Polyominos

Composition

Coding words

- Freeman's alphabet : $\mathcal{F} = \{0, 1, 2, 3\}$.

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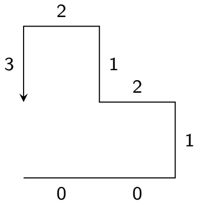
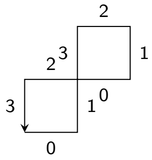
Example

The word 0012123 codes



Boundary word

Closed as many 0s than 2s, 1s than 3s.



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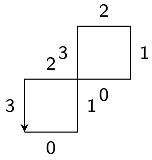
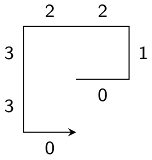


Self avoiding no strict factor is closed.



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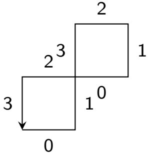

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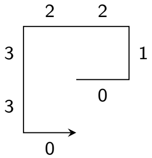
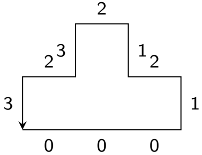
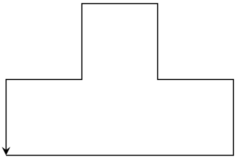


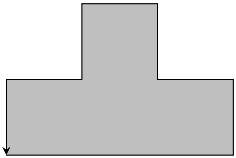
Boundary closed and self avoiding.



Polyomino



Polyomino



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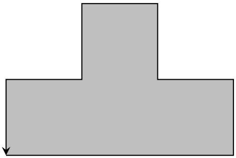
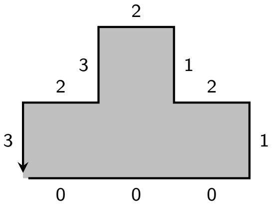


Definition

It is the topological closure of the inside of the loop coded by a boundary word.

Polyomino



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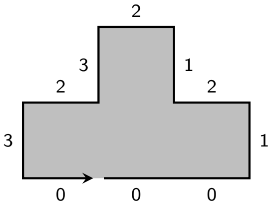

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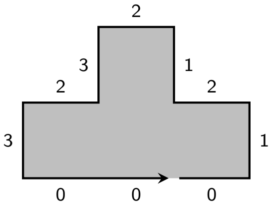

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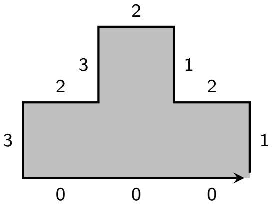

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Tesselation

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A tesselation is a covering of $\mathbb{R}^2$ by translations of a unique polyomino overlapping with empty interior.
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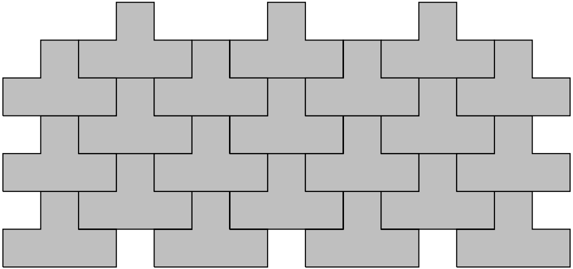

Beauquier-Nivat

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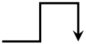
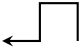
Example

The hat of 0103 () is 1232 ().

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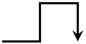
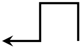
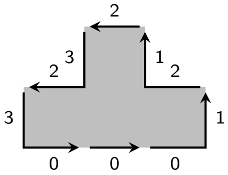
The hat of 0103 () is 1232 ().

Beauquier-Nivat's Theorem

A polyomino is exact if and only if it admits a boundary word of the form $uvw\hat{u}\hat{v}\hat{w}$.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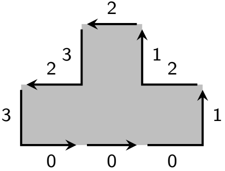
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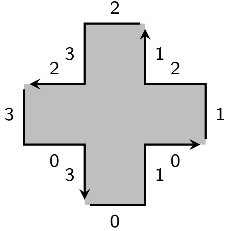
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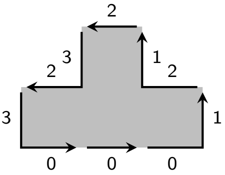
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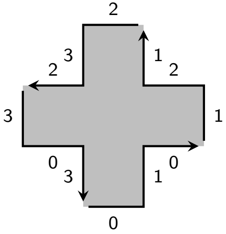
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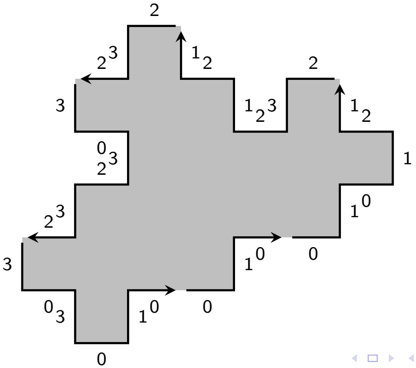
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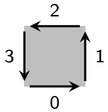
Polyominos

Composition

Square tiles

Unit square

Boundary word 0123.



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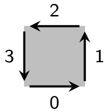
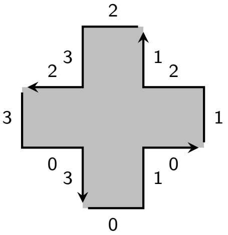


Square

Boundary word of the form $uv\hat{u}\hat{v}$.



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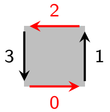

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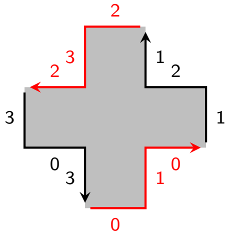

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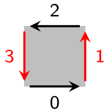

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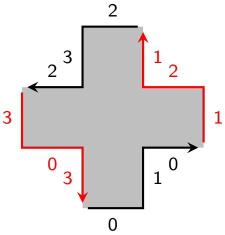

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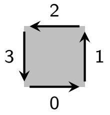

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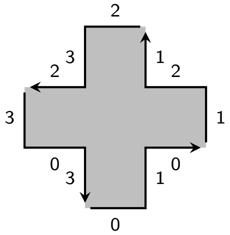
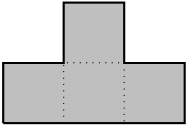


Composition

Geometric intuition

Idea

Replace the unit square by a square tile.



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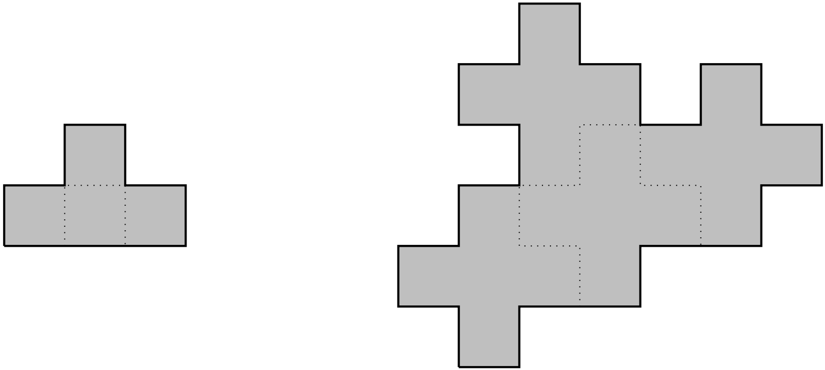

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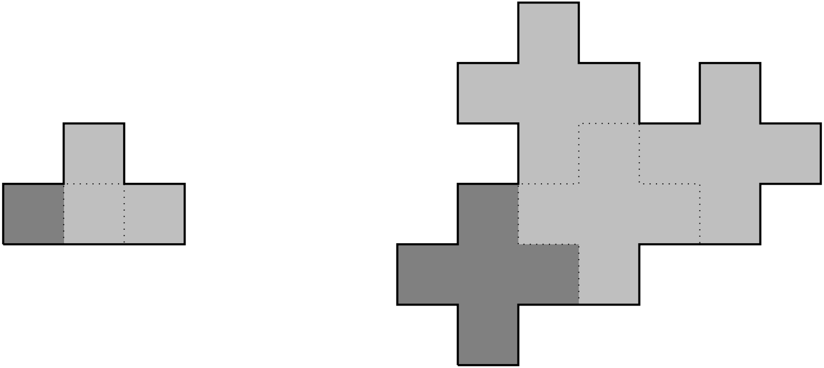

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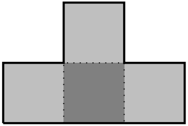

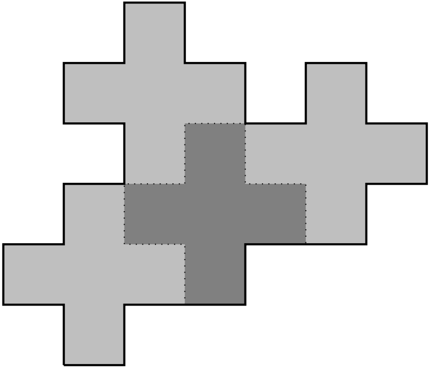

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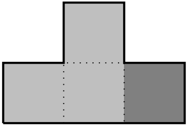

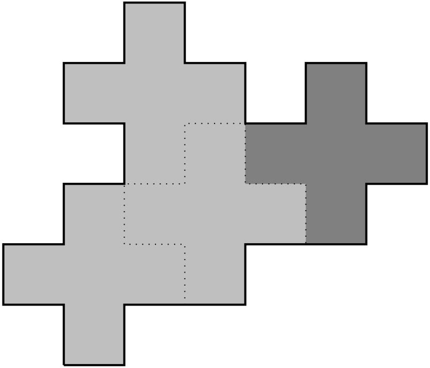

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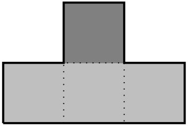

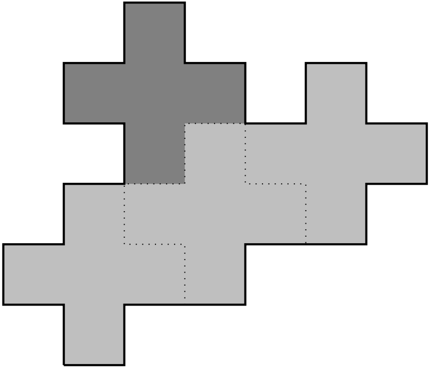

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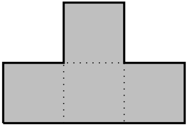

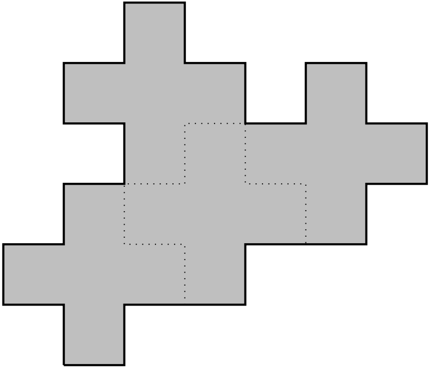

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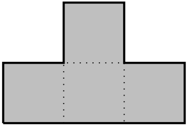

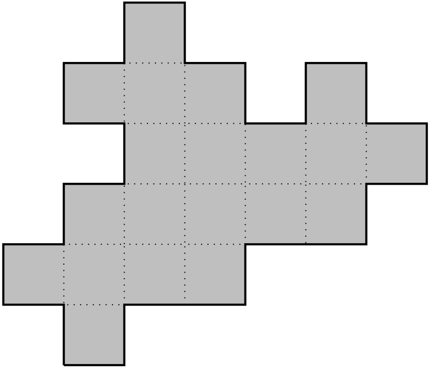

Composition

Formal definition

Definition

Let u, v such that $uv\hat{u}\hat{v}$ is a boundary word. The *composition by* (u, v) is the monoid morphism

$$\varphi_{u,v}: \quad 0 \mapsto u, \quad 1 \mapsto v, \quad 2 \mapsto \hat{u}, \quad 3 \mapsto \hat{v}.$$

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Previous example :

$$\varphi_{010,121}(0001212323) = 010010010121232121232303232303.$$

Question

If w codes an **exact** polyomino, does $\varphi_{u,v}(w)$ as well ?

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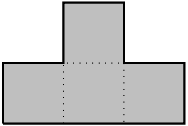

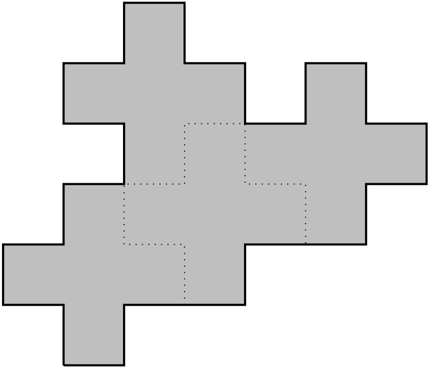

Sufficiency

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Proof. $\varphi_{u,v}(\hat{\alpha}) = \widehat{\varphi_{u,v}(\alpha)}$.

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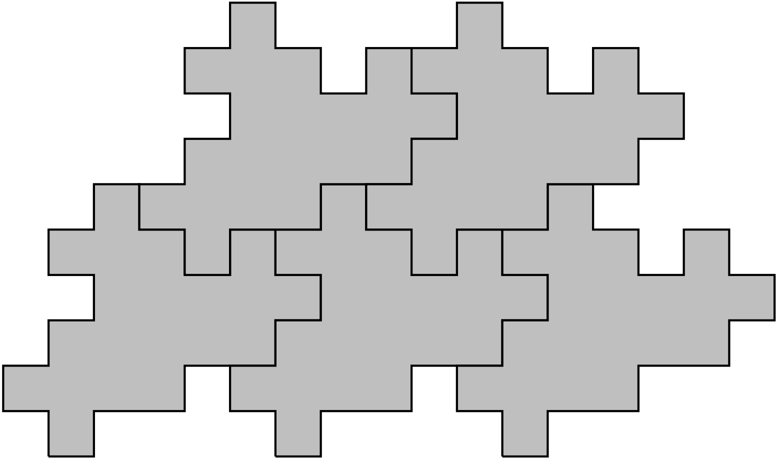
Necessity

If $\varphi_{u,v}(w)$ codes an exact polyomino, does w also ?

Proof. $\varphi_{u,v}(w) = ABC\widehat{A}\widehat{B}\widehat{C}...$

Idea



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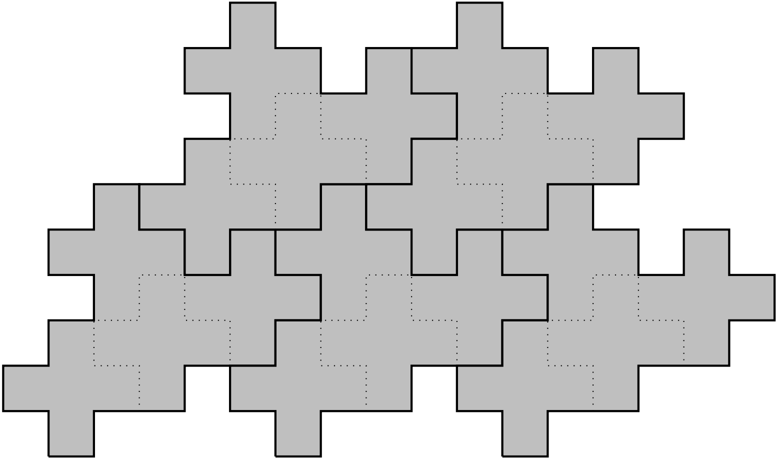

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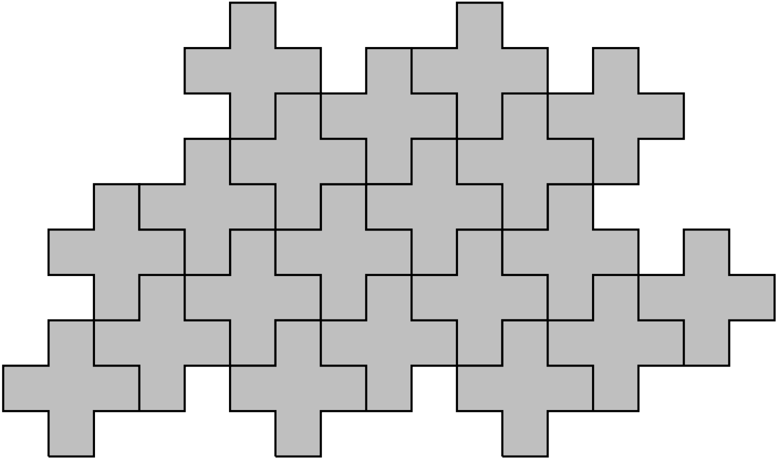
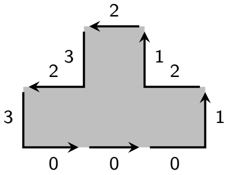


Beauquier-Nivat

2.0

Theorem

An exact polyomino has a boundary word of the form $uvw\hat{u}\hat{v}\hat{w}$.



Beauquier-Nivat

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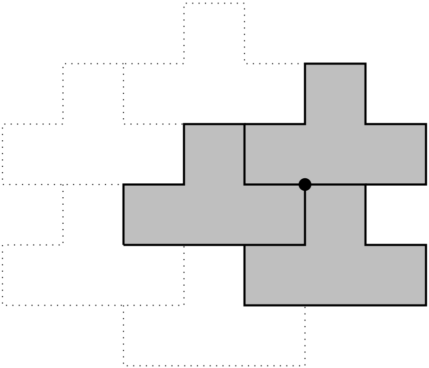
Theorem

Every *meeting point* of a tesselation gives a boundary word of the form $uvw\hat{u}\hat{v}\hat{w}$.



Beauquier-Nivat

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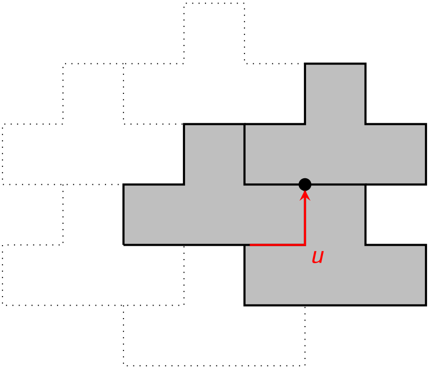

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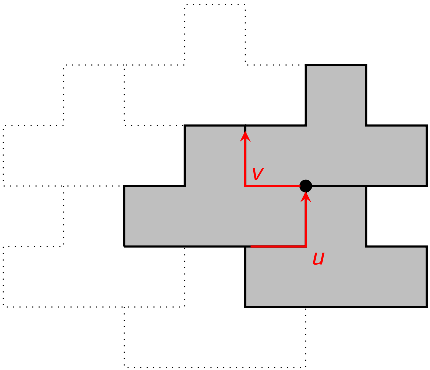

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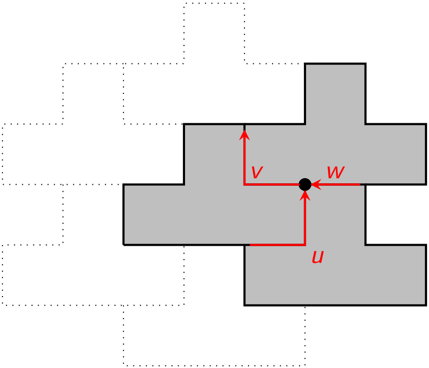

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- **General case ?** The previous proof does not extend...

Thanks.

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